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A quantum algebra approach to basic multivariable special functions

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Abstract. Generalizations to a base q of the Appell and Lauricella hypergeometric functions are studied within the framework of the representation theory of quantum algebras of type A_n. This allows deriving various identities and formulae involving these q-functions.

1. Introduction

Quantum algebras and groups constitute very powerful algebraic tools for studying many physical models, in particular those that involve discrete space and/or time. The literature dealing with their applications is by now very large, and ranges over many different fields, from mathematical physics to condensed matter physics.‖

More specifically, quantum algebras are deformations with a complex parameter q of the universal enveloping algebras of classical Lie algebras [4, 5]. Like their classical counterparts, they often arise in physical models as symmetry algebras, simplifying the analysis of the dynamics of these models and allowing in many instances for a complete solution [6–8].

It is well known that the representation theory of the classical Lie algebras gives a unifying algebraic setting for many special functions of mathematical physics [9–12]. Quantum algebras play a similar role for generalizations to a base q of these functions, the so-called q-special functions [13]. Though unfamiliar to most physicists before the advent of quantum groups, the q-functions are now playing a fundamental role in the study of quantum algebra symmetries. In fact, these functions naturally arise whenever such structures are relevant to the description of physical systems; in particular, n-point correlation functions can be expressed in terms of q-hypergeometric functions in n-variables [14–18].

Motivated by this, here we shall study multivariable q-special functions within the quantum algebra framework. We shall provide an interpretation of these functions along the lines developed in the case of the single variable basic hypergeometric functions [19–35]. Using this approach, various identities and properties for these functions will be derived.

After a section where the notations and definitions used throughout the paper are introduced, we present in section 3 results on the representation theory of $\mathcal{U}_q(sl(n+3))$. In the following section, we study a q-generalization of the Appell function, a two-variable

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‖ For instance, see the various contributions in [1–3].

version of the Heine hypergeometric series, and its connection with the quantum algebra $\mathcal{U}_q(sl(5))$. The q-Appell functions appear both in matrix elements of certain operators in $\mathcal{U}_q(sl(5))$ and also as basis vectors, in specific irreducible representations of subalgebras of $\mathcal{U}_q(sl(5))$. Section 5 is devoted to an n-variable generalization of the Heine hypergeometric series, the extension to a base q of the Lauricella function F_D. In analogy to the q-Appell series, this function proves to be connected with the representation theory of $\mathcal{U}_q(sl(n+3))$. Only the results are given in this case, since the derivations are similar to those involving the q-Appell functions.

Let us stress that the techniques described here are essentially independent of convergence criteria. The relations and formulae involving the q-Appell and q-Lauricella functions obtained in sections 4 and 5 should be looked at as identities between formal power series; it could happen that these converge over a finite radius or only when the series terminate.

2. Notation

We start with a few formulae in q-analysis that will be used in the following. In our approach, an important role is played by the following two q-exponential functions [13]:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n = \frac{1}{(z;q)_\infty} \qquad \text{for } |z| < 1 \tag{2.1a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n = (-z;q)_\infty \tag{2.1b}$$

where, for a and α arbitrary complex numbers, $(a;q)_\alpha$ stands for the q-shifted factorial

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{2.2}$$

with

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad \text{for } |q| < 1. \tag{2.3}$$

Note that $e_q(z)\,E_q(-z) = 1$, and that $\lim_{q\to 1^-} e_q\big(z(1-q)\big) = \lim_{q\to 1^-} E_q\big(z(1-q)\big) = e^z$. We shall denote by T_z the q-dilatation operator which acts as

$$T_z\,\varphi(z) = \varphi(qz) \tag{2.4}$$

on functions of the variable z; out of it, the q-difference operators

$$D_z^+ = z^{-1}(1 - T_z) \tag{2.5a}$$

$$D_z^- = z^{-1}(1 - T_z^{-1}) \tag{2.5b}$$

are constructed. Observe that $\frac{1}{(1-q)} D_z^+ \to \mathrm{d}/\mathrm{d}z$ and $\frac{1}{(1-q^{-1})} D_z^- \to \mathrm{d}/\mathrm{d}z$ as $q \to 1$. Notice also that the q-exponentials obey

$$D_z^+\,e_q(\lambda z) = \lambda e_q(\lambda z) \tag{2.6a}$$

$$D_z^-\,E_q(\lambda z) = -q^{-1}\lambda E_q(\lambda z) \tag{2.6b}$$

with λ a complex parameter.

The basic hypergeometric series ${}_r\phi_s$ is defined [13] by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) &\equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \ldots (a_r; q)_n}{(q; q)_n (b_1; q)_n \ldots (b_s; q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}} \right]^{1+s-r} z^n \end{aligned} \tag{2.7}$$

with $q \neq 0$ when $r > s+1$. Since $(q^{-m}; q)_n = 0$, for $n = m+1, m+2, \ldots$, the series ${}_r\phi_s$ terminates if one of the numerator parameters $\{a_i\}$ is of the form q^{-m} with $m = 0, 1, 2\ldots$ and $q \neq 0$. By the ratio test, when $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \leqslant s$, and for $|z| < 1$ if $r = s+1$. The particular case $r = 2$ and $s = 1$ gives the Heine q-series

$$ {}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n \qquad \text{for } |z| < 1 \tag{2.8}$$

which in the limit $q \to 1^-$ reduces to the standard Gauss hypergeometric series.

In the following we shall consider multivariable versions of the functions defined in (2.7) and (2.8). One of the simplest generalizations of the Heine q-series is the basic Appell series Φ_D [36]:

$$\Phi_D(a; b, b'; c; q; x, y) = \sum_{m,n=0}^{\infty} \frac{(a; q)_{m+n} (b; q)_m (b'; q)_n}{(c; q)_{m+n} (q; q)_m (q; q)_n} x^m y^n. \tag{2.9}$$

For $0 < |q| < 1$, it converges absolutely when $|x| < 1$ and $|y| < 1$.

The generalization to n-variables of (2.9) gives the basic Lauricella function Φ_D [37, 38]:

$$\begin{aligned} &\Phi_D(a; b_1, \ldots, b_n; c; q; z_1, \ldots, z_n) \\ &\qquad = \sum_{m_1, \ldots, m_n \geqslant 0} \frac{(a; q)_M (b_1; q)_{m_1} \cdots (b_n; q)_{m_n}}{(c; q)_M (q; q)_{m_1} \cdots (q; q)_{m_n}} z_1^{m_1} \ldots z_n^{m_n} \end{aligned} \tag{2.10}$$

where $M = m_1 + m_2 + \ldots + m_n$. Also in this case the series converges absolutely when $|z_i| < 1$, $i = 1, 2, \ldots, n$, for $0 < |q| < 1$. In the following we shall fix q to be in this range. It is easy to check that the q-Lauricella function (2.10) satisfies the following q-difference equations

$$\begin{aligned} &\left[(1 - c T_z) D_{z_i}^{+} - (1 - a T_z)(1 - b_i T_{z_i}) \right] \Phi_D = 0 \qquad \text{for } 1 \leqslant i \leqslant n \\ &\left[(1 - b_i T_{z_i}) D_{z_j}^{+} - (1 - b_j T_{z_j}) D_{z_i}^{+} \right] \Phi_D = 0 \qquad \text{for } 1 \leqslant i < j \leqslant n \end{aligned} \tag{2.11}$$

where $T_z = T_{z_1} T_{z_2} \cdots T_{z_n}$. Also notice that Φ_D can be expressed in terms of a ${}_{n+1}\phi_n$ q-hypergeometric function [37, 38]:

$$\begin{aligned} &\Phi_D(a; b_1, \ldots, b_n; c; q; z_1, \ldots, z_n) = \\ &\qquad \frac{(a; q)_\infty}{(c; q)_\infty} \prod_{i=1}^{n} \left[\frac{(b_i z_i; q)_\infty}{(z_i; q)_\infty} \right] {}_{n+1}\phi_n \left[\begin{matrix} c/a, & z_1, & \ldots, & z_n \\ & b_1 z_1, & \ldots, & b_n z_n \end{matrix} ; q, a \right]. \end{aligned} \tag{2.12}$$

We shall see that the series (2.10) arises in the representation theory of the quantum algebra $\mathcal{U}_q(sl(n+3))$.

3. The quantum algebra $\mathcal{U}_q(sl(n+3))$

The quantum universal enveloping algebra $\mathcal{U}_q(sl(n+3))$ is the Hopf algebra generated by the elements k_i, k_i^{-1}, e_i and f_i, $i = 1, 2, \ldots, n+2$, satisfying the defining relations [5]

$$
\begin{aligned}
&k_i\, k_i^{-1} = k_i^{-1}\, k_i = 1 \qquad k_i\, k_j = k_j\, k_i \qquad && k_i\, e_j\, k_i^{-1} = q^{a_{ij}}\, e_j \\
&[e_i, f_j] = \delta_{ij} \frac{k_i^{1/2} - k_i^{-1/2}}{q^{1/2} - q^{-1/2}} && k_i\, f_j\, k_i^{-1} = q^{-a_{ij}}\, f_j
\end{aligned}
\tag{3.1}
$$

together with the Serre-like relations

$$
\begin{aligned}
&[e_i, e_j] = 0 = [f_i, f_j] \qquad \text{for } |i-j| > 1 \\
&e_i^2\, e_{i\pm1} - (q^{1/2} + q^{-1/2})\, e_i\, e_{i\pm1}\, e_i + e_{i\pm1}\, e_i^2 = 0 \\
&f_i^2\, f_{i\pm1} - (q^{1/2} + q^{-1/2})\, f_i\, f_{i\pm1}\, f_i + f_{i\pm1}\, f_i^2 = 0
\end{aligned}
\tag{3.2}
$$

where a_{ij} is the Cartan matrix of type A_{n+2}, with $a_{ii} = 2$, $a_{ii\pm1} = -1$ and $a_{ij} = 0$ otherwise. Its irreducible representations can by studied by introducing the following abstract operators $\mathcal{E}$ acting on the basis functions $f_{\alpha,\beta_1,\ldots,\beta_n,\gamma} \equiv f_{\alpha,\beta_i,\gamma}$, with $\alpha, \beta_1, \ldots, \beta_n, \gamma$ complex numbers [39, 40]:

$$
\begin{aligned}
\mathcal{E}^{\alpha}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\gamma-\alpha-1})\, f_{\alpha+1,\beta_i,\gamma} \\
\mathcal{E}_{\alpha}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\alpha-1})\, f_{\alpha-1,\beta_i,\gamma} \\
\mathcal{E}^{\beta_k}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta_k})\, f_{\alpha,\hat{\beta}_k,\gamma} \\
\mathcal{E}_{\beta_k}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\gamma-\beta})\, f_{\alpha,\tilde{\beta}_k,\gamma} \\
\mathcal{E}^{\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta-\gamma})\, f_{\alpha,\beta_i,\gamma+1} \\
\mathcal{E}_{\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\gamma-\alpha-1})\, f_{\alpha,\beta_i,\gamma-1} \\
\mathcal{E}^{\alpha\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta-\gamma})\, f_{\alpha+1,\beta_i,\gamma+1} \\
\mathcal{E}_{\alpha\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\alpha-1})\, f_{\alpha-1,\beta_i,\gamma-1} \\
\mathcal{E}^{\beta_k\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta_k})\, f_{\alpha,\hat{\beta}_k,\gamma+1} \\
\mathcal{E}_{\beta_k\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\gamma-\beta})\, f_{\alpha,\tilde{\beta}_k,\gamma-1} \\
\mathcal{E}^{\beta_j}_{\beta_k}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta_j})\, f_{\alpha,\beta_1,\ldots,\beta_j+1,\ldots,\beta_k-1,\ldots,\beta_n,\gamma} \\
\mathcal{E}^{\alpha\beta_k\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{\beta_k})\, f_{\alpha+1,\hat{\beta}_k,\gamma+1} \\
\mathcal{E}_{\alpha\beta_k\gamma}\, f_{\alpha,\beta_i,\gamma} &= (1 - q^{1-\alpha})\, f_{\alpha-1,\tilde{\beta}_k,\gamma-1} \\
k_{\alpha}\, f_{\alpha,\beta_i,\gamma} &= q^{\alpha}\, f_{\alpha,\beta_i,\gamma} \\
k_{\beta_k}\, f_{\alpha,\beta_i,\gamma} &= q^{\beta_k}\, f_{\alpha,\beta_i,\gamma} \\
k_{\gamma}\, f_{\alpha,\beta_i,\gamma} &= q^{\gamma}\, f_{\alpha,\beta_i,\gamma}
\end{aligned}
\tag{3.3}
$$

where $\beta = \sum_{k=0}^{n} \beta_k$, $\hat{\beta}_k = \beta_1, \ldots, \beta_k + 1, \ldots, \beta_n$ and $\tilde{\beta}_k = \beta_1, \ldots, \beta_k - 1, \ldots, \beta_n$. For simplicity, let us also define

$$
k_{\beta} = k_{\beta_1}\, k_{\beta_2} \cdots k_{\beta_n}
\tag{3.4}
$$

so that

$$
k_{\beta}\, f_{\alpha,\beta_i,\gamma} = q^{\beta}\, f_{\alpha,\beta_i,\gamma}.
\tag{3.5}
$$

The operators $\mathcal{E}$ generate the quantum algebra $\mathcal{U}_q(sl(n+3))$. Indeed, with the following redefinitions:

$$\begin{aligned}
e_{n+2} &= \frac{k_\alpha^{-1/4}\,k_\beta^{1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}^\gamma & f_{n+2} &= \frac{k_\alpha^{3/4}\,k_\beta^{-3/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}_\gamma & k_{n+2} &= q^{-1}\,k_\alpha^{-1}\,k_\beta^{-1}\,k_\gamma^2 \\
e_{n+1} &= \frac{k_\alpha^{-3/4}\,k_\beta^{-3/4}\,k_\gamma^{3/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}_{\alpha\gamma} & f_{n+1} &= \frac{k_\alpha^{1/4}\,k_\beta^{1/4}\,k_\gamma^{-1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}^{\alpha\gamma} & k_{n+1} &= q\,k_\alpha^{-1}\,k_\beta\,k_\gamma^{-1} \\
e_n &= \frac{k_\alpha^{1/4}\,k_{\beta_n}^{-1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}^{\alpha\beta_n\gamma} & f_n &= \frac{k_\alpha^{1/4}\,k_{\beta_n}^{-1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}_{\alpha\beta_n\gamma} & k_n &= q^{-1}\,k_\alpha\,k_{\beta_n} \\
e_i &= \frac{q^{1/4}\,k_{\beta_i}^{-1/4}\,k_{\beta_{i+1}}^{-1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}_{\beta_{i+1}}^{\beta_i} & f_i &= \frac{q^{1/4}\,k_{\beta_i}^{-1/4}\,k_{\beta_{i+1}}^{-1/4}}{q^{1/2}-q^{-1/2}}\,\mathcal{E}_{\beta_i}^{\beta_{i+1}} & k_i &= k_{\beta_i}\,k_{\beta_{i+1}}^{-1} \\
& & & & i &= 1,\ldots,n-1
\end{aligned} \tag{3.6}$$

the relations (3.1) and (3.2) are verified. The operators $\mathcal{E}$ listed in (3.3) and not appearing in (3.6) correspond to non-simple roots.

Let $(\alpha^0, \beta_i^0, \gamma^0)$, $i = 1, 2, \ldots, n$, be fixed complex numbers, not integers, and let $\alpha = \alpha^0 + m$, $\beta_i = \beta_i^0 + n_i$, $\gamma = \gamma^0 + k$, where (m, n_i, k) run over all integers. Then the basis functions $\{f_{\alpha,\beta_i,\gamma}\}$ and the operators $\mathcal{E}$ in (3.3) define an infinite-dimensional irreducible representation of $\mathcal{U}_q(sl(n+3))$. In the following sections we shall see how this representation gives a natural algebraic setting for the q-Appell and q-Lauricella functions.

Let us finally point out that it is easy to construct an explicit realization of the above representation, using $n+2$ complex variables: (x, y_i, z), $i = 1, 2, \ldots, n$. In this model, the basis functions are

$$f_{\alpha,\beta_i,\gamma}(x, y_1, \ldots, y_n, z) = x^\alpha y_1^{\beta_1} \cdots y_n^{\beta_n} z^\gamma \tag{3.7}$$

while the elements $\mathcal{E}$ are expressed in terms of q-difference operators

$$\begin{aligned}
\mathcal{E}^\alpha &= x\,(1 - q^{-1} T_x^{-1}\,T_z) & \mathcal{E}_\alpha &= \frac{1}{x}\,(1 - q^{-1} T_x) \\
\mathcal{E}^{\beta_k} &= y_k\,(1 - T_{y_k}) & \mathcal{E}_{\beta_k} &= \frac{1}{y_k}\,(1 - T_y^{-1}\,T_z) \\
\mathcal{E}^\gamma &= z\,(1 - T_y\,T_z^{-1}) & \mathcal{E}_\gamma &= \frac{1}{z}\,(1 - q^{-1} T_x^{-1}\,T_z) \\
\mathcal{E}^{\alpha\gamma} &= xz\,(1 - T_y\,T_z^{-1}) & \mathcal{E}_{\alpha\gamma} &= \frac{1}{xz}\,(1 - q^{-1} T_x) \\
\mathcal{E}^{\beta_k\gamma} &= y_k z\,(1 - T_{y_k}) & \mathcal{E}_{\beta_k\gamma} &= \frac{1}{y_k z}\,(1 - T_y^{-1}\,T_z) \\
\mathcal{E}^{\alpha\beta_k\gamma} &= x y_k z\,(1 - T_{y_k}) & \mathcal{E}_{\alpha\beta_k\gamma} &= \frac{1}{x y_k z}\,(1 - q\,T_x^{-1}) \\
\mathcal{E}_{\beta_j}^{\beta_k} &= \frac{y_k}{y_j}\,(1 - T_{y_k}) & k_\alpha &= T_x \\
k_{\beta_k} &= T_{y_k} & k_\gamma &= T_z
\end{aligned} \tag{3.8}$$

where $T_y = T_{y_1} T_{y_2} \cdots T_{y_n}$. One can easily check that the operators (3.8) acting on the monomials (3.7) verify the relations (3.3).

4. The basic Appell function and $\mathcal{U}_q(sl(5))$

In this section we shall fix $n = 2$ and consider the irreducible representation (3.3) for $\mathcal{U}_q(sl(5))$. The algebraic interpretation of the q-Appell function $\Phi_D(a; b, b'; c; q; x, y)$ that we shall present is based on a suitable subalgebra of $\mathcal{U}_q(sl(5))$. Let us consider the following operators: $\mathcal{E}^\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}^\gamma$, $\mathcal{E}^{\alpha\gamma}$. Together with the elements k_α and k_γ, they satisfy the following relations:

$$
\begin{aligned}
&[\mathcal{E}^\alpha, \mathcal{E}_\alpha] = q^{-1}(1-q)(k_\alpha^{-1} k_\gamma - k_\alpha) && [\mathcal{E}_\alpha, \mathcal{E}^{\alpha\gamma}] = q^{-1}(1-q)\, k_\alpha\, \mathcal{E}^\gamma \\
&[\mathcal{E}^\alpha, \mathcal{E}^\gamma] = q^{-1}(1-q) k_\alpha^{-1} k_\gamma\, \mathcal{E}^{\alpha\gamma} && [\mathcal{E}^\alpha, \mathcal{E}^{\alpha\gamma}] = 0 \\
&[\mathcal{E}_\alpha, \mathcal{E}^\gamma] = 0 && [\mathcal{E}^\gamma \mathcal{E}^{\alpha\gamma}] = 0 \\
&k_\alpha\, \mathcal{E}^\alpha = q\, \mathcal{E}^\alpha\, k_\alpha && k_\alpha\, \mathcal{E}_\alpha = q^{-1}\, \mathcal{E}_\alpha\, k_\alpha \\
&k_\alpha\, \mathcal{E}^\gamma = \mathcal{E}^\gamma\, k_\alpha && k_\alpha\, \mathcal{E}^{\alpha\gamma} = q\, \mathcal{E}^{\alpha\gamma}\, k_\alpha \\
&k_\gamma\, \mathcal{E}^\alpha = \mathcal{E}^\alpha\, k_\gamma && k_\gamma\, \mathcal{E}_\alpha = \mathcal{E}_\alpha\, k_\gamma \\
&k_\gamma\, \mathcal{E}^\gamma = q\, \mathcal{E}^\gamma\, k_\gamma && k_\gamma\, \mathcal{E}^{\alpha\gamma} = q\, \mathcal{E}^{\alpha\gamma}\, k_\gamma .
\end{aligned}
\tag{4.1}
$$

Other choices of subalgebras are possible; a different one will be presented in the next section.

In the completion of $\mathcal{U}_q(sl(5))$, let us now consider the following operator:

$$
U(a, b, c, d) = E_q(a\, \mathcal{E}^\alpha)\, E_q(b\, \mathcal{E}^\gamma k_\alpha)\, E_q(c\, \mathcal{E}^{\alpha\gamma})\, e_q(d\, \mathcal{E}_\alpha). \tag{4.2}
$$

Notice that in the limit $q \to 1^-$, this operator reduces to an element of the group $SL(5)$. The matrix elements of $U(a, b, c, d)$ on the basis functions $\{f_{\alpha,\beta_1,\beta_2,\gamma}\}$ of the representation (3.3) can be computed explicitly using the definition (2.1) and various identities involving q-shifted factorials. They are defined in the usual way:

$$
U(a, b, c, d)\, f_{\alpha,\beta_1,\beta_2,\gamma} = \sum_{\alpha',\gamma'} f_{\alpha',\beta_1,\beta_2,\gamma'}\, U_{\alpha'\gamma',\alpha\gamma}(a, b, c, d) \tag{4.3}
$$

where $(\alpha', \alpha) \in \alpha^0 + \mathbb{Z}$, $(\gamma', \gamma) \in \gamma^0 + \mathbb{Z}$, as discussed in the previous section. Notice that the operators $\mathcal{E}^\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}^\gamma$, $\mathcal{E}^{\alpha\gamma}$ do not change the indices (β_1, β_2) of the basis vectors $f_{\alpha,\beta_1,\beta_2,\gamma}$. This means that the matrix elements of $U(a, b, c, d)$ are non-zero only for $\beta_1' = \beta_1$ and $\beta_2' = \beta_2$; for simplicity, we have suppressed the fixed indices (β_1, β_2) in their definition. The explicit computation gives

$$
\begin{aligned}
U_{\alpha'\gamma',\alpha\gamma}(a, b, c, d) &= q^{(\gamma'-\gamma)(\gamma'-\gamma-1+2\alpha')/2} \frac{(q; q)_{\alpha-1}}{(q; q)_{\alpha'-1}} \frac{(q; q)_{\beta_1+\beta_2-\gamma}}{(q; q)_{\beta_1+\beta_2-\gamma'}} \frac{b^{\gamma'-\gamma}}{(q; q)_{\gamma'-\gamma}} \frac{d^{\alpha-\alpha'}}{(q; q)_{\alpha-\alpha'}} \\
&\quad \times \Phi_D\Big(q^{1-\alpha'}; q^{\gamma'-\alpha'}, q^{\gamma-\gamma'}; q^{\alpha-\alpha'+1}; q; -ad\, q^{\alpha'+\gamma-\gamma'-1}, cd/b\Big) \\
&\qquad \text{if } \gamma' - \gamma \geqslant 0,\ \alpha - \alpha' \geqslant 0
\end{aligned}
\tag{4.4a}
$$

$$U_{\alpha'\gamma',\alpha\gamma}(a,b,c,d) = q^{(\alpha'-\alpha)(\alpha'-\alpha-1)/2+(\gamma-\gamma')(\gamma-\gamma'+\alpha'-\alpha)} \times \frac{(q;q)_{\gamma-\alpha-1}}{(q;q)_{\gamma'-\alpha'-1}} \frac{(q;q)_{\beta_1+\beta_2-\gamma}}{(q;q)_{\beta_1+\beta_2-\gamma'}} \frac{a^{\alpha'-\alpha+\gamma-\gamma'}}{(q;q)_{\alpha'-\alpha+\gamma-\gamma'}} \frac{c^{\gamma'-\gamma}}{(q;q)_{\gamma'-\gamma}} \times \Phi_D\Big(q^{\gamma-\alpha}; q^{1-\alpha}, q^{\gamma-\gamma'}; q^{\alpha'-\alpha+\gamma-\gamma'+1}; q; -ad\,q^{\alpha'+\gamma-\gamma'-1}, -q^{\alpha'}ab/c\Big) \quad \text{if } \gamma'-\gamma \geqslant 0,\ \alpha'-\alpha \geqslant 0. \tag{4.4b}$$

This establishes in a simple way the connection between the basic Appell function Φ_D and $\mathcal{U}_q(sl(5))$. Notice that the two formulae above are valid irrespective of the sign of $\alpha' - \alpha$, thanks to the following limiting formula (m and n positive integers):

$$\frac{1}{(q;q)_{-m}} \Phi_D(a; b, q^{-n}; q^{1-m}; q; x, y) = q^{-n(n+1)/2} x^{m-n} (-y)^n \frac{(a;q)_m\,(b;q)_{m-n}}{(q;q)_{m-n}} \times \Phi_D(bq^{m-n}; aq^m, q^{-n}; q^{m-n+1}; q; x, q^{n+1}x/y). \tag{4.5}$$

Other combinations of q-exponentials involving the generators $\mathcal{E}^\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}^\gamma$, $\mathcal{E}^{\alpha\gamma}$ can also be used. Of particular interest is

$$\widetilde{U}(a,b,c,d) = E_q(d\,\mathcal{E}_\alpha)\, e_q(c\,\mathcal{E}^{\alpha\gamma})\, e_q(b\,\mathcal{E}^\gamma k_\alpha)\, e_q(a\,\mathcal{E}^\alpha). \tag{4.6}$$

The matrix elements of $\widetilde{U}(a,b,c,d)$ in the representation (3.3) are given by

$$\widetilde{U}_{\alpha'\gamma',\alpha\gamma}(a,b,c,d) = q^{(\alpha-\alpha')(\alpha-\alpha'-1)/2+\alpha(\gamma'-\gamma)} \frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}} \frac{(q;q)_{\beta_1+\beta_2-\gamma}}{(q;q)_{\beta_1+\beta_2-\gamma'}} \frac{b^{\gamma'-\gamma}}{(q;q)_{\gamma'-\gamma}} \frac{d^{\alpha-\alpha'}}{(q;q)_{\alpha-\alpha'}} \times \Phi_D\Big(q^\alpha; q^{\alpha-\gamma+1}, q^{\gamma-\gamma'}; q^{\alpha-\alpha'+1}; q; -ad\,q^{\gamma'-\alpha'-1}, -q^{\gamma'-\gamma-\alpha'}cd/b\Big) \quad \text{if } \gamma'-\gamma \geqslant 0,\ \alpha-\alpha' \geqslant 0 \tag{4.7a}$$

$$\widetilde{U}_{\alpha'\gamma',\alpha\gamma}(a,b,c,d) = \frac{(q;q)_{\gamma-\alpha-1}}{(q;q)_{\gamma'-\alpha'-1}} \frac{(q;q)_{\beta_1+\beta_2-\gamma}}{(q;q)_{\beta_1+\beta_2-\gamma'}} \frac{a^{\alpha'-\alpha+\gamma-\gamma'}}{(q;q)_{\alpha'-\alpha+\gamma-\gamma'}} \frac{c^{\gamma'-\gamma}}{(q;q)_{\gamma'-\gamma}} \times \Phi_D\Big(q^{\alpha'-\gamma'+1}; q^{\alpha'}, q^{\gamma-\gamma'}; q^{\alpha'-\alpha+\gamma-\gamma'+1}; q; -ad\,q^{\gamma'-\alpha'-1}, q^{\gamma'}ab/c\Big) \quad \text{if } \gamma'-\gamma \geqslant 0,\ \alpha'-\alpha \geqslant 0. \tag{4.7b}$$

A biorthogonality relation involving two q-Appell functions can be derived by combining the results (4.4) and (4.7). Recalling that $e_q(z)\,E_q(-z) = 1$, one sees that

$$U(a,b,c,d)\,\widetilde{U}(-a,-b,-c,-d) = \mathbb{1} \tag{4.8}$$

or alternatively, acting on the basis vector $f_{\alpha,\beta_1,\beta_2,\gamma}$, that

$$\sum_{\overline{\alpha},\overline{\gamma}} U_{\alpha'\gamma',\overline{\alpha}\overline{\gamma}}(a,b,c,d)\; \widetilde{U}_{\overline{\alpha}\overline{\gamma},\alpha\gamma}(-a,-b,-c,-d) = \delta_{\alpha'-\alpha,0}\,\delta_{\gamma'-\gamma,0} \tag{4.9}$$

with $\overline{\alpha} \in \alpha^0 + \mathbb{Z}$, $\overline{\gamma} \in \gamma^0 + \mathbb{Z}$, and $\gamma' - \overline{\gamma} \geqslant 0$, $\overline{\gamma} - \gamma \geqslant 0$. Substituting for the matrix elements $U_{\alpha'\gamma',\overline{\alpha}\overline{\gamma}}(a,b,c,d)$ and $\widetilde{U}_{\overline{\alpha}\overline{\gamma},\alpha\gamma}(-a,-b,-c,-d)$, the expressions (4.4$a$) and (4.7$a$), after some simplifications and redefinitions, one arrives at the following formula ($\overline{\gamma} - \gamma = m \in \mathbb{Z}^+$, $\alpha - \overline{\alpha} = k \in \mathbb{Z}$):

$$\delta_{\alpha'-\alpha,0}\,\delta_{\gamma'-\gamma,0} = \sum_{m,k} q^{(m+k)(\alpha-\alpha')+m} \frac{(q^{\gamma-\gamma'};q)_m}{(q;q)_m} \frac{(q^{\alpha'-\alpha};q)_k}{(q;q)_k} \times \Phi_D\Big(q^{1-\alpha'}; q^{\gamma'-\alpha'}, q^{\gamma-\gamma'+m}; q^{\alpha-\alpha'-k+1}; q; x\,q^m, y\Big) \times \Phi_D\Big(q^\alpha; q^{\alpha-\gamma+1}, q^{-m}; q^{k+1}; q; x\,q^{\gamma'-\alpha'-\alpha+m+k}, y\,q^{m+k-\alpha}\Big). \tag{4.10}$$

Another relation involving the series Φ_D can be obtained by using the model (3.8) for the representation (3.3). In the present case, the basis functions $f_{\alpha\beta_1\beta_2\gamma}$ are expressed by the monomials $x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma$, while the operators $\mathcal{E}^\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}^\gamma$, $\mathcal{E}^{\alpha\gamma}$ and k_α are as in (3.8), with $T_y = T_{y_1} T_{y_2}$. Let us now act directly with the operator $U(a,b,c,d)$ on $x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma$. By using the definition (2.1) of the q-exponentials and with the help of Heine's q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(q^{-\alpha};q)_n}{(q;q)_n} z^n = \frac{(zq^{-\alpha};q)_\infty}{(z;q)_\infty} = (zq^{-\alpha};q)_\alpha \qquad |z|<1, \quad |q|<1 \tag{4.11}$$

one finds that

$$\begin{aligned} U(a,b,c,d)\, x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma &= x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma\, (-ax;q)_{\gamma-\alpha-1}\, (-xzc;q)_{\beta_1+\beta_2-\gamma} \\ &\quad \times \sum_{m=0}^{\infty} q^{-m(m-1)/2} \frac{(q^{1-\alpha};q)_m\, (-axq^{\gamma-\alpha-1};q)_m}{(q;q)_m} \left(-q^{\alpha-1} d/x\right)^m \\ &\quad \times {}_2\phi_1\Big(-ax\, q^{\gamma-\alpha+m-1}, q^{\gamma-\beta_1-\beta_2}; -xzc; q, -bz\, q^{\alpha+\beta_1+\beta_2-\gamma-m}\Big). \end{aligned} \tag{4.12}$$

To proceed further, we use the following integral representation for the series ${}_2\phi_1$:

$${}_2\phi_1(a,b;c;q,z) = \frac{(a,b;q)_\infty}{(c,q;q)_\infty} \left(\frac{\mathrm{i}}{2\pi}\right) \int_C \frac{(q^{s+1}, cq^s;q)_\infty}{(aq^s, bq^s;q)_\infty} \frac{\pi(-z)^s}{\sin(\pi s)}\, \mathrm{d}s \tag{4.13}$$

where we have used the standard notation

$$(a_1, a_2, \ldots, a_n; q)_\alpha = (a_1;q)_\alpha\, (a_2;q)_\alpha \cdots (a_n;q)_\alpha \tag{4.14}$$

and where C is a suitable generalization of Barnes' contour in the complex s-plane (for details, see [13]). By exchanging the sum with the integral and using once more equation (4.11), one finally arrives at the formula

$$\begin{aligned} U(a,b,c,d)\, x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma &= x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma\, \frac{(q^{\gamma-\beta_1-\beta_2}, -ax;q)_\infty}{(-xzcq^{\beta_1+\beta_2-\gamma}, q;q)_\infty} \\ &\quad \times \left(\frac{\mathrm{i}}{2\pi}\right) \int_C \frac{(q^{s+1}, -xzc\,q^s;q)_\infty}{(q^{s+\gamma-\beta_1-\beta_2}, -axq^{s+\gamma-\alpha-1};q)_\infty} \frac{\pi}{\sin(\pi s)} \\ &\quad \times \left(bz\, q^{\alpha+\beta_1+\beta_2-\gamma}\right)^s {}_2\phi_0\Big(q^{1-\alpha}, -ax\, q^{s+\gamma-\alpha-1}; q, -q^{\alpha-s-1} d/x\Big)\, \mathrm{d}s. \end{aligned} \tag{4.15}$$

Since the series ${}_2\phi_0(a,b;q,z)$ does not converge, unless it terminates or $z=0$, the action of $U(a,b,c,d)$ on $x^\alpha y_1^{\beta_1} y_2^{\beta_2} z^\gamma$ is ill-defined in this model, unless α is an integer greater than unity. Nevertheless, by proceeding formally one can obtain an interesting identity for the q-hypergeometric series Φ_D. Recall the definition (4.3) for the matrix elements of $U(a,b,c,d)$; in this formula insert (4.15) in the LHS, and substitute (4.4a) for $U_{\alpha'\gamma',\alpha\gamma}(a,b,c,d)$ in the RHS. With suitable manipulations and obvious redefinitions, one finally ends up with the following relation involving the q-Appell function:

$$\begin{aligned} &\frac{(c, -y/z;q)_\infty}{(-xt/qz, q;q)_\infty} \left(\frac{\mathrm{i}}{2\pi}\right) \\ &\qquad \times \int_C \frac{(q^{s+1}, -q^{s-1}xtc/z;q)_\infty}{(cq^s, -q^{s-1}by/z;q)_\infty} \frac{\pi(-t)^s}{\sin(\pi s)}\, {}_2\phi_0(a, -q^{s-1}by/z; q, zq^{-s})\, \mathrm{d}s \\ &\qquad = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} q^{-k(k+2m-1)/2} \frac{(a;q)_k}{(q;q)_k} \frac{(c;q)_m}{(q;q)_m} z^k t^m \\ &\qquad \times \Phi_D\Big(aq^k; q^{-m}, bq^{m+k}; q^{k+1}; q; x, yq^{-m-k}\Big). \end{aligned} \tag{4.16}$$

Further identities for the function Φ_D can be obtained by noting that the matrix elements $U_{\alpha'\gamma',\alpha\gamma}(a,b,c,d)$ and $\widetilde{U}_{\alpha'\gamma',\alpha\gamma}(a,b,c,d)$ themselves provide models for $\mathcal{U}_q(sl(5))$ modules. Indeed, one can check that the following q-difference operators in the four complex variables a, b, c and d, depending parametrically on the complex numbers α', γ'

$$\pi^{(\alpha',\gamma')}(k_\alpha) = q^{\alpha'} T_a^{-1} T_c^{-1} T_d \tag{4.17a}$$

$$\pi^{(\alpha',\gamma')}(k_\gamma) = q^{\gamma'} T_b^{-1} T_c^{-1} \tag{4.17b}$$

$$\pi^{(\alpha',\gamma')}(\mathcal{E}_\alpha) = D_d^+ \tag{4.17c}$$

$$\pi^{(\alpha',\gamma')}(\mathcal{E}^\gamma) = -q^{1-\alpha'} T_a D_b^- \tag{4.17d}$$

$$\pi^{(\alpha',\gamma')}(\mathcal{E}^{\alpha\gamma}) = -q D_c^- - d T_c^{-1} D_b^- \tag{4.17e}$$

$$\pi^{(\alpha',\gamma')}(\mathcal{E}^\alpha) = -q D_a^- - (d/q)\left(q^{\gamma'-\alpha'} T_a T_b^{-1} T_d^{-1} - q^{\alpha'} T_a^{-1} T_c^{-1}\right) + q^{\alpha'} b T_a^{-1} D_c^- \tag{4.17f}$$

acting on the basis functions

$$f^{(\alpha',\gamma')}_{\alpha\beta_1\beta_2\gamma}(a,b,c,d) = U_{\alpha'\gamma',\alpha\gamma}(a,b,c,d) \tag{4.18}$$

verify the algebra (4.1). (For a hint on how the formulae (4.17) are obtained, see [26] and the following section.) From this realization one can get useful formulae for the function $\Phi_D(a;b,b';c;q;x,y)$. In fact, by acting with the operators (4.17c–f) on (4.18), one immediately derives

$$\begin{aligned}
&(1-(c/q)\,T_xT_y)\,\Phi_D = (1-c/q)\,\Phi_D(c/q)\\
&(1-b'T_y)\,\Phi_D = (1-b')\,\Phi_D(qb')\\
&(qD_y^- + T_y^{-1} - b')\,T_x^{-1}\,\Phi_D = \frac{(1-b')(a-c)}{(1-c)}\,\Phi_D(qb',qc)\\
&\left[(qb'/a)D_x^- + (1/a)\,T_x^{-1}T_y^{-1} + (q/a)\,T_x^{-1}D_y^- - (bb'/c)\right]\Phi_D\\
&\qquad\qquad = \frac{(1-bb'/c)(1-c/a)}{(1-c)}\,\Phi_D(qc)
\end{aligned} \tag{4.19}$$

where Φ_D stands for $\Phi_D(a;b,b';c;q;x,y)$, and $\Phi_D(qa)$ for $\Phi_D(aq;b,b';c;q;x,y)$, and so on. Further relations can be obtained from the explicit definition (2.9) of the series Φ_D, or by starting with a subalgebra of $\mathcal{U}_q(sl(5))$ different from that of (4.1):

$$\begin{aligned}
&(1-a\,T_xT_y)\,\Phi_D = (1-a)\,\Phi_D(qa)\\
&(1-b\,T_x)\,\Phi_D = (1-b)\,\Phi_D(qb)\\
&D_x^+\,\Phi_D = \frac{(1-a)(1-b)}{(1-c)}\,\Phi_D(qa,qb,qc)\\
&\left[(1-(c/q)\,T_xT_y) - (bx/q)(1-a\,T_xT_y)\right]\Phi_D = (1-c/q)\,\Phi_D(b/q,c/q)\\
&\left[(1-(c/a)\,T_xT_y) - (x/q)(1-b\,T_x)T_y - (y/q)(1-b'\,T_y)\right]T_x^{-1}T_y^{-1}\,\Phi_D\\
&\qquad\qquad = (1-c/a)\,\Phi_D(a/q).
\end{aligned} \tag{4.20}$$

These relations together with those in (4.19) constitute a complete set of 'contiguity' relations for the q-Appell functions. In particular, it follows that this function satisfies the following second-order q-difference equations:

$$\begin{aligned}
&\Big[(1 - c\,T_xT_y)\,D_x^+ - (1 - a\,T_xT_y)(1 - b\,T_x)\Big]\,\Phi_D(a;b,b';c;q;x,y) = 0 \\
&\Big[(1 - c\,T_xT_y)D_y^+ - (1 - a\,T_xT_y)(1 - b'T_x)\Big]\,\Phi_D(a;b,b';c;q;x,y) = 0 \\
&\Big[(1 - b\,T_x)\,D_y^+ - (1 - b'\,T_y)D_x^+\Big]\,\Phi_D(a;b,b';c;q;x,y) = 0.
\end{aligned} \tag{4.21}$$

The four-variable model (4.17), (4.18) can be further used to get generating and addition formulae for the q-Appell functions. From the general definition (4.3), one can write, recalling (4.18),

$$\widetilde{U}(a',b',c',d')\,f^{(\alpha',\gamma')}_{\alpha\beta_1\beta_2\gamma}(a,b,c,d) = \sum_{\overline{\alpha},\overline{\gamma}} U_{\alpha'\gamma',\overline{\alpha}\overline{\gamma}}(a,b,c,d)\,\widetilde{U}_{\overline{\alpha}\overline{\gamma},\alpha\gamma}(a',b',c',d') \tag{4.22}$$

where the model-independent matrix elements $\widetilde{U}_{\overline{\alpha}\overline{\gamma},\alpha\gamma}(a',b',c',d')$ are still given by (4.7) and $\overline{\alpha} \in \alpha^0 + \mathbb{Z}$, $\overline{\gamma} \in \gamma^0 + \mathbb{Z}$, $\overline{\gamma} - \gamma \geqslant 0$, $\gamma' - \overline{\gamma} \geqslant 0$. To derive useful identities from (4.22), one needs to evaluate the LHS of this equation, i.e. to compute directly the action of $\widetilde{U}(a',b',c',d')$ on the basis functions (4.18), using the realization (4.17).

We shall start by considering the simple case in which only the parameter d' is non-vanishing, $\widetilde{U}(d') = E_q(d'\,\mathcal{E}_\alpha)$. One can check that, in this case, the only non-zero matrix elements are

$$\widetilde{U}_{\alpha'\gamma,\alpha\gamma}(d') = q^{(\alpha-\alpha')(\alpha-\alpha'-1)/2}\,\frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}}\,\frac{(d')^{\alpha-\alpha'}}{(q;q)_{\alpha-\alpha'}} \qquad \alpha - \alpha' \geqslant 0. \tag{4.23}$$

The action of $E_q(d'\,D_d^+)$ on $f^{(\alpha',\gamma')}(a,b,c,d)$ can be computed by using the following summation formula ($|d'/d| < 1$):

$$E_q(d'\,D_d^+)\,d^n = d^n\,(-d'/d;q)_n. \tag{4.24}$$

For $d' = -qd$, this action can be rewritten again in terms of a Φ_D:

$$\begin{aligned}
E_q(-qd\,D_d^+)\,f^{(\alpha',\gamma')}_{\alpha\beta_1\beta_2\gamma}(a,b,c,d) &= q^{(\gamma'-\gamma)(\gamma'-\gamma-1+2\alpha')/2}\,\frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}}\,\frac{(q;q)_{\beta_1+\beta_2-\gamma}}{(q;q)_{\beta_1+\beta_2-\gamma'}} \\
&\quad\times \frac{b^{\gamma'-\gamma}}{(q;q)_{\gamma'-\gamma}}\,d^{\alpha-\alpha'}\,\Phi_D\Big(q^{1-\alpha'};q^{\gamma'-\alpha'},q^{\gamma-\gamma'};0;q;-ad\,q^{\alpha'+\gamma-\gamma'-1},cd/b\Big).
\end{aligned} \tag{4.25}$$

Inserting this result, together with the explicit expressions (4.4a) and (4.23) for the matrix elements $U_{\alpha'\gamma',\overline{\alpha}\overline{\gamma}}(a,b,c,d)$ and $\widetilde{U}_{\overline{\alpha}\overline{\gamma},\alpha\gamma}(-dq)$ in (4.22), with obvious manipulations and redefinitions one finally obtains

$$(q;q)_\alpha\,\Phi_D(a;b,b';0;q;x,y) = \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}\begin{bmatrix}\alpha\\ m\end{bmatrix}_q\,\Phi_D(a;b,b';q^{\alpha-m+1};q;x,y) \tag{4.26}$$

where

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q;q)_\alpha}{(q;q)_\beta\,(q;q)_{\alpha-\beta}} \tag{4.27}$$

is the q-binomial symbol.

To get an addition formula from (4.22), one needs to consider the general case where all the parameters a', b', c' and d' are non-zero. The action of $\widetilde{U}(a',b',c',d')$ on $f^{(\alpha',\gamma')}(a,b,c,d)$ can not be summed in general, but only when the following conditions are satisfied: $b' = -c'd'/q$, $d' = -qd$, and $b = cd/q$. In this case one can prove that

$$\begin{aligned}\widetilde{U}(a',c'd,c',-qd)\, f^{(\alpha',\gamma')}_{\alpha\beta_1\beta_2\gamma}(a,cd/q,c,d) &= q^{(\gamma'-\gamma)(\alpha'-1)}\frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}} \\ &\times \frac{(q;q)_{\beta_1+\beta_2-\gamma}}{(q;q)_{\beta_1+\beta_2-\gamma'}}\frac{(-c/c';q)_{\gamma'-\gamma}}{(q;q)_{\gamma'-\gamma}} d^{\alpha-\alpha'}\,(c'd)^{\gamma'-\gamma} \\ &\times {}_2\phi_0\Big(q^\alpha, q^{\alpha-\gamma+1}; q, a'd\, q^{\gamma-\alpha-1}\Big) \\ &\times \Phi_D\Big(q^{1-\alpha'}; q^{\gamma'-\alpha'}, q^{\gamma-\gamma'}; 0; q; -ad\, q^{\alpha'+\gamma-\gamma'-1}, q\Big).\end{aligned} \tag{4.28}$$

In deriving this result, use has been made of the q-binomial summation formula (4.11) and of the q-Gauss' summation theorem:

$${}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \tag{4.29}$$

With the help of the explicit expressions (4.4a) and (4.7a) of the matrix elements $U_{\alpha'\gamma',\overline{\alpha\gamma}}(a,cd/q,c,d)$ and $\widetilde{U}_{\overline{\alpha\gamma},\alpha\gamma}(a',c'd,c',-qd)$, after suitable operations, equation (4.22) finally becomes

$$\begin{aligned}(q;q)_{\alpha-\alpha'}\,&(zq^{\gamma-\gamma'};q)_{\gamma'-\gamma}\,{}_2\phi_0\Big(q^\alpha, q^{\alpha-\gamma+1}; q, y/q\Big)\ \Phi_D\Big(q^{1-\alpha'}; q^{\gamma'-\alpha'}, q^{\gamma-\gamma'}; 0; x, q\Big) \\ &= \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} q^{(m+k)(\alpha-\alpha'+1)}\frac{(q^{\gamma-\gamma'};q)_m\,(q^{\alpha'-\alpha};q)_k}{(q;q)_m\,(q;q)_k} z^m \\ &\quad\times \Phi_D\Big(q^{1-\alpha'}; q^{\gamma'-\alpha'}, q^{\gamma-\gamma'+m}; q^{\alpha-\alpha'-k+1}; q; xq^m, q\Big) \\ &\quad\times \Phi_D\Big(q^\alpha; q^{\alpha-\gamma+1}, q^{-m}; q^{k+1}; q; y\,q^{m+k}, q^{m+k-\alpha+1}\Big).\end{aligned} \tag{4.30}$$

This is a simple addition formula for the q-Appell function. Concerning the convergence of (4.30), the remarks made after (4.15) also apply; in particular, for ${}_2\phi_0$ to be convergent, α must be a negative integer.

5. The basic Lauricella function and $\mathcal{U}_q(sl(n+3))$

We now generalize the considerations of the previous section to the case of q-hypergeometric series in many variables. Though the strategy is essentially unchanged, the computations are much more involved. For this reason, we shall limit the exposition to the results giving only indications on how these are derived.

The starting point is the quantum algebra $\mathcal{U}_q(sl(n+3))$ in the realization given in (3.3). In this case also, we shall concentrate on a suitable subalgebra generated by the elements $\mathcal{E}^\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}^{\alpha\beta_i\gamma}$, $\mathcal{E}^{\beta_i\gamma}$, k_α and k_{β_i}, $i=1,2,\ldots,n-1$.

In the completion of $\mathcal{U}_q(sl(n+3))$, consider then the following operator:

$$U(a,b_1,\ldots,b_{n-1},c_1,\ldots,c_{n-1},d) = e_q(a\,\mathcal{E}^\alpha)\prod_{i=1}^{n-1}\left[e_q(b_i\,\mathcal{E}^{\alpha\beta_i\gamma})\right]\prod_{j=1}^{n-1}\left[E_q(c_j\,\mathcal{E}^{\beta_j\gamma})\right]e_q(d\,\mathcal{E}_\alpha k_\alpha^{-1}) \tag{5.1}$$

which in the limit $q\to 1^-$ becomes an element of the group $SL(n+3)$. In the following we shall use a shorthand notation and simply write $U(a,b_i,c_i,d)$ for the RHS of (5.1). The matrix elements of $U(a,b_i,c_i,d)$ with respect to the basis vectors $\{f_{\alpha\beta_i\gamma}\}$ can be computed using (3.3). These are zero, unless the new index γ' is equal to $\gamma+\sum_j(\beta_j'-\beta_j)$; for this reason we shall simply denote them as $U_{\alpha'\beta_i',\alpha\beta_i}(a,b_i,c_i,d)$, the index γ being understood. Explicitly one finds ($\beta_i'-\beta_i\geqslant 0$)

$$\begin{aligned}U_{\alpha'\beta_i',\alpha\beta_i}(a,b_i,c_i,d) &= q^{(\alpha'-\alpha)(\alpha'+\alpha+1)/2}\frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}}\frac{d^{\alpha-\alpha'}}{(q;q)_{\alpha-\alpha'}}\\ &\times\prod_{j=1}^{n-1}\left[q^{(\beta_j'-\beta_j)(\beta_j'-\beta_j-1)/2}(q^{\beta_j};q)_{\beta_j'-\beta_j}\frac{c_j^{\beta_j'-\beta_j}}{(q;q)_{\beta_j'-\beta_j}}\right]\\ &\times\Phi_D\left(q^{1-\alpha'};q^{\gamma-\alpha'+\sum_j(\beta_j'-\beta_j)},q^{\beta_i-\beta_i'};q^{\alpha-\alpha'+1};q;-ad/q,b_id/c_i\right)\end{aligned} \tag{5.2}$$

where $(\alpha',\alpha)\in\alpha^0+\mathbb{Z}$, $(\beta_i',\beta_i)\in\beta_i^0+\mathbb{Z}$ and $\gamma\in\gamma^0+\mathbb{Z}$, $i=1,\ldots,n-1$. In this expression, $\Phi_D(a;b,b_i;c;q;x,y_i)$ stands for the n-variable q-Lauricella function $\Phi_D(a;b,b_1,\ldots,b_{n-1};\ c;q;x,y_1,\ldots,y_{n-1})$, which is thus directly connected with the representation theory of $\mathcal{U}_q(sl(n+3))$. The form (5.2) of the matrix elements, though derived under the assumption $\alpha-\alpha'\geqslant 0$, is valid irrespective from the sign of $\alpha-\alpha'$, thanks to the following limiting relation (m and k_i, $i=1,\ldots,n-1$, positive integers)

$$\begin{aligned}\frac{1}{(q;q)_{-m}}\Phi_D(a;b,q^{-k_i};q^{1-m};q;x,y_i) &= \frac{(a;q)_m\,(b;q)_{m-K}}{(q;q)_{m-K}}x^m\\ &\times\prod_{j=1}^{n-1}\left[q^{-k_j(k_j+1)/2}(-y_j/x)^{k_i}\right]\\ &\times\Phi_D(bq^{m-K};aq^m,q^{-k_i};q^{m-K+1};q;x,q^{k_i+1}x/y_i)\end{aligned} \tag{5.3}$$

where $K=\sum_{j=1}^{n-1}k_j$. This relation clearly reduces to (4.5) for $n=2$. Similarly, one can check that the operator

$$\widetilde{U}(a,b_i,c_i,d) = E_q(d\,\mathcal{E}_\alpha k_\alpha^{-1})\prod_{i=1}^{n-1}\left[e_q(c_i\,\mathcal{E}^{\beta_i\gamma})\right]\prod_{j=1}^{n-1}\left[E_q(b_j\,\mathcal{E}^{\alpha\beta_j\gamma})\right]E_q(a\,\mathcal{E}^\alpha) \tag{5.4}$$

has the following matrix elements ($\beta_i'-\beta_i\geqslant 0$)

$$\begin{aligned}\widetilde{U}_{\alpha'\beta_i',\alpha\beta_i}(a,b_i,c_i,d) &= q^{(\alpha'-\alpha)(\alpha'+1)}\frac{(q;q)_{\alpha-1}}{(q;q)_{\alpha'-1}}\frac{d^{\alpha-\alpha'}}{(q;q)_{\alpha-\alpha'}}\prod_{j=1}^{n-1}\left[(q^{\beta_j};q)_{\beta_j'-\beta_j}\frac{c_j^{\beta_j'-\beta_j}}{(q;q)_{\beta_j'-\beta_j}}\right]\\ &\times\Phi_D\Big(q^\alpha;q^{\alpha-\gamma+1},q^{\beta_i-\beta_i'};q^{\alpha-\alpha'+1};q;\\ &\qquad -ad\,q^{\gamma-\alpha-\alpha'-2},-q^{\beta_i'-\beta_i-\alpha'-1}b_id/c_i\Big)\end{aligned} \tag{5.5}$$

for $\gamma' = \gamma + \sum_j (\beta'_j - \beta_j)$, and zero otherwise.

The operator (5.4) has been chosen in such a way that

$$U(a, b_i, c_i, d)\, \widetilde{U}(-a, -b_i, -c_i, -d) = \mathbf{1}\,. \tag{5.6}$$

Applying the basis vector $f_{\alpha\beta_i\gamma}$ to both sides of this equation, one finds

$$\sum_{\overline{\alpha}\overline{\beta}_i} U_{\alpha'\beta'_i,\overline{\alpha}\overline{\beta}_i}(a, b_i, c_i, d)\ \widetilde{U}_{\overline{\alpha}\overline{\beta}_i,\alpha\beta_i}(-a, -b_i, -c_i, -d) = \delta_{\alpha'-\alpha,0} \prod_{j=1}^{n-1} \delta_{\beta'_j-\beta_j,0}. \tag{5.7}$$

This gives a biorthogonality relation for the q-Lauricella function. In fact, using the explicit expressions (5.2) and (5.5) for the matrix elements, with suitable redefinitions one gets

$$\begin{aligned}\delta_{\alpha'-\alpha,0} \prod_{j=1}^{n-1} \delta_{\beta'_j-\beta_j,0} &= \sum_{k=-\infty}^{\infty} q^{k(\alpha-\alpha')} \frac{(q^{\alpha'-\alpha}; q)_k}{(q;q)_k} \prod_{j=1}^{n-1} \left[\sum_{m_j=0}^{\infty} q^{m_j} \frac{(q^{\beta_j-\beta'_j}; q)_{m_j}}{(q;q)_{m_j}} \right] \\ &\times \Phi_D\Big(q^{1-\alpha'}; q^{\gamma-\alpha'+\sum_j(\beta'_j-\beta_j)}, q^{\beta_i-\beta'_i+m_i}; q^{\alpha-\alpha'-k+1}; q; x, y_i\Big) \\ &\times \Phi_D\Big(q^{\alpha}; q^{\alpha-\gamma+1}, q^{-m_i}; q^{k+1}; q; x q^{\gamma-2\alpha+k-1}, y_i\, q^{k+m_i-\alpha-1}\Big).\end{aligned} \tag{5.8}$$

In analogy to what has been done in the previous section, one can use the realization (3.8) of the algebra $\mathcal{U}_q(sl(n+3))$ to get a generating relation for the q-Lauricella function. Consider the operator $E_q(a\mathcal{E}^{\alpha}) \prod_{i=1}^{n-1} \left[e_q(b_i \mathcal{E}^{\beta_i\gamma} k_\alpha) e_q(c_i \mathcal{E}^{\alpha\beta_i\gamma}) \right] e_q(d\mathcal{E}_\alpha)$, whose matrix elements also involve the Φ_D function. Its action on the monomials (3.7) can be expressed in terms of a ${}_{n+1}\phi_{n-1}$ hypergeometric function, by using the summation formulae (4.11) and (4.29). This allows deriving the following generating formula:

$$\begin{aligned}&\frac{(x;q)_{\gamma-1}(xz;q)_\gamma}{(q^{1-\alpha};q)_\alpha \prod_{j=1}^{n-1}(y_j;q)_{\beta_j}}\ {}_{n+1}\phi_{n-1}\left[\begin{matrix} q^{1-\alpha}, xq^{\gamma-1}, & y_1, & \ldots, & y_{n-1} \\ & q^{\beta_1}y_1, & \ldots, & q^{\beta_{n-1}}y_{n-1} \end{matrix}; q, z\right] \\ &\quad = \sum_{k=-\infty}^{\infty} q^{-k(k+1)/2}(-1/z)^k (xz;q)_k \\ &\qquad \times \Phi_D\Big(xzq^k; q^\alpha, q^{\beta_i}; xzq^\gamma; q; q^{1-\alpha-k}, y_i q^{-k}\Big).\end{aligned} \tag{5.9}$$

As in the case of the q-Appell function, further identities involving the q-Lauricella function can be derived by using a representation for the algebra $\mathcal{U}_q(sl(n+3))$ in which these functions appear as basis vectors. A complete description of this representation in terms of quantum Grassmannians has been given in [41]. A similar model for $\mathcal{U}_q(sl(n+3))$ can also be derived using the algebraic approach to the basic Lauricella functions described above. As an illustration, we shall derive below representatives of the elements k_α, $\mathcal{E}_\alpha$, $\mathcal{E}^{\beta_i\gamma}$ and $\mathcal{E}^{\alpha\beta_i\gamma}$ taking the matrix elements (5.2) as basis vectors.

Using the relations (3.3), it is easy to check that $k_\alpha \mathcal{E}^\alpha k_\alpha^{-1} = q\mathcal{E}^\alpha$, $k_\alpha \mathcal{E}_\alpha k_\alpha^{-1} = q^{-1}\mathcal{E}_\alpha$, $k_\alpha \mathcal{E}^{\alpha\beta_i\gamma} k_\alpha^{-1} = q\mathcal{E}^{\alpha\beta_i\gamma}$ and $k_\alpha \mathcal{E}^{\beta_i\gamma} k_\alpha^{-1} = \mathcal{E}^{\beta_i\gamma}$. From the definition (5.1), it then follows that

$$k_\alpha\, U(a, b_i, c_i, d)\, k_\alpha^{-1} = U(qa, qb_i, c_i, q^{-1}d). \tag{5.10}$$

Acting with both sides of this formula on $f_{\alpha\beta_i\gamma}$ one gets

$$q^{\alpha'} T_a^{-1}\, T_b^{-1}\, T_d\, U_{\alpha'\beta',\alpha\beta}(a, b_i, c_i, d) = q^{\alpha}\, U_{\alpha'\beta',\alpha\beta}(a, b_i, c_i, d) \tag{5.11}$$

where $T_b = T_{b_1} T_{b_2} \cdots T_{b_{n-1}}$ and $U_{\alpha'\beta',\alpha\beta}(a, b_i, c_i, d)$ are the matrix elements in (5.2). In other words, the operator

$$\pi^{(\alpha',\beta'_j)}(k_\alpha) = q^{\alpha'} T_a^{-1} T_b^{-1} T_d \tag{5.12}$$

acting on the basis functions

$$f^{(\alpha',\beta'_i)}_{\alpha\beta_i\gamma}(a, b_i, c_i, d) = U_{\alpha'\beta',\alpha\beta_i}(a, b_i, c_i, d) \tag{5.13}$$

depending parametrically on α', β'_i and γ, represents the element k_α of $\mathcal{U}_q(sl(n+3))$.

Analogously, recalling (2.6a), one has

$$D_d^+ U(a, b_i, c_i, d) = U(a, b_i, c_i, d)\, \mathcal{E}_\alpha\, k_\alpha^{-1}. \tag{5.14}$$

Using the result (5.11), one immediately finds

$$\pi^{(\alpha',\beta'_j)}(\mathcal{E}_\alpha) = q^{\alpha'+1} T_a^{-1} T_b^{-1} T_d\, D_d^+. \tag{5.15}$$

Similarly, from (2.6b) and $[\mathcal{E}_\alpha, \mathcal{E}^{\beta_i\gamma}] = 0$, one derives

$$\pi^{(\alpha',\beta'_j)}(\mathcal{E}^{\beta_i\gamma}) = -q\, D_{c_i}^-. \tag{5.16}$$

To get $\pi^{(\alpha',\beta'_j)}(\mathcal{E}^{\alpha\beta_i\gamma})$, one first acts with $D_{b_i}^+$ on $U(a, b_i, c_i, d)$ to get

$$D_{b_i}^+ U(a, b_i, c_i, d) = U(a, b_i, c_i, d)\, E_q(-d\, \mathcal{E}_\alpha k_\alpha^{-1})\, \mathcal{E}^{\alpha\beta_i\gamma}\, e_q(d\, \mathcal{E}_\alpha k_\alpha^{-1}) \tag{5.17}$$

and then shows that

$$E_q(-d\, \mathcal{E}_\alpha k_\alpha^{-1})\, \mathcal{E}^{\alpha\beta_i\gamma}\, e_q(d\, \mathcal{E}_\alpha k_\alpha^{-1}) = \mathcal{E}^{\alpha\beta_i\gamma} - (d/q)\, \mathcal{E}^{\beta_i\gamma}\, k_\alpha^{-1} \tag{5.18}$$

with the help of the following relation:

$$E_q(-\lambda X)\; Y\, e_q(\lambda X) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{(q;q)_n} [X, Y]_n \tag{5.19}$$

where

$$[X, Y]_0 = Y \qquad [X, Y]_{n+1} = q^n X [X, Y]_n - [X, Y]_n X \qquad \text{for } n = 1, 2, \ldots\,.$$

At the end one finds

$$\pi^{(\alpha',\beta'_j)}(\mathcal{E}^{\alpha\beta_i\gamma}) = D_{b_i}^+ - q^{-\alpha'} d\, T_a T_b T_d^{-1}\, D_{c_i}^-. \tag{5.20}$$

Let us now act with the operators (5.15), (5.16) and (5.20) on the basis vectors (5.13); recalling the explicit expression for the matrix elements $U_{\alpha'\beta',\alpha\beta}(a, b_i, c_i, d)$ in terms of the q-Lauricella function, one obtains the following identities:

$$\begin{aligned}
&(1 - (c/q)\, T_z)\, \Phi_D = (1 - c/q)\, \Phi_D(c/q) \\
&(1 - b_i\, T_{z_i})\, \Phi_D = (1 - b_i)\, \Phi_D(qb_i) \\
&\Big[(1 - b_i\, T_{z_i}) - (c/a) D_{z_i}^+\Big]\, \Phi_D = \frac{(1 - b_i)(1 - c/a)}{(1 - c)}\, \Phi_D(qb_i, qc)
\end{aligned} \tag{5.21}$$

where $T_z = T_{z_1} T_{z_2} \cdots T_{z_n}$, and Φ_D stands for $\Phi_D(a; b_1, \ldots, b_n; c; q; z_1, \ldots, z_n)$, while $\Phi_D(c/q)$ for $\Phi_D(a; b_1, \ldots, b_n; c/q; q; z_1, \ldots, z_n)$, and so on. These relations together with

$$\begin{aligned} D^+_{z_i} \Phi_D &= \frac{(1-a)(1-b_i)}{(1-c)} \Phi_D(qa, qb_i, qc) \\ (1 - a\,T_z)\,\Phi_D &= (1-a)\,\Phi_D(qa) \end{aligned} \tag{5.22}$$

an immediate consequence of the definition (2.10), imply the q-difference equations (2.11).

A simple generating formula for the q-Lauricella function can now be derived using again (5.15) and (5.16), following the same steps that led us to (4.26). Take the operator $\widetilde{U}(a', b'_i, c'_i, d')$ and set $a' = 0$, $b'_i = 0$, $i = 1, \ldots, n-1$. By means of the summation formula (4.24) and

$$\begin{aligned} e_q(-qc'_i D^-_{c_i})\, c_i^n &= c_i^n \frac{1}{(-qc'_i/c_i; q)_{-n}} \\ &= q^{-n(n-1)/2} (-c_i/c'_i; q)_n\, (c'_i)^n \qquad \text{for } |c'_i/c_i| < 1 \end{aligned} \tag{5.23}$$

and of the transformation relation (2.12), the action of $U(0, 0, c'_i, d')$ on the basis functions (5.13) can be expressed in terms of a q-hypergeometric function ${}_{n+1}\phi_n$, provided $d' = -qd$ and $c'_i = -c_i/q$. This result together with (5.2) and (5.5), when inserted in

$$\widetilde{U}(0, 0, -c_i/q, -qd)\, f^{(\alpha', \beta'_i)}_{\alpha\beta_i\gamma} = \sum_{\overline{\alpha}\overline{\beta}_i} U_{\alpha'\beta', \overline{\alpha}\overline{\beta}_i}(a, b_i, c_i, d)\, \widetilde{U}_{\overline{\alpha}\overline{\beta}_i, \alpha\beta_i}(0, 0, -c_i/q, -qd) \tag{5.24}$$

gives the following relation

$$\begin{aligned} &(q; q)_\alpha \frac{(a; q)_\infty (bx; q)_\infty}{(x; q)_\infty} \\ &\qquad \times \prod_{j=1}^{n-1} \left[(-1)^{\beta_j} q^{-\beta_j(\beta_j+1)/2} \frac{(q; q)_{\beta_j}}{(y_j; q)_\infty} \right] {}_{n+1}\phi_n \left[\begin{matrix} 0, \ x, \ y_1, \ \ldots, \ y_{n-1} \\ bx, \ 0, \ \ldots, \ 0 \end{matrix} ; q, a \right] \\ &\quad = \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} \begin{bmatrix} \alpha \\ m \end{bmatrix}_q \prod_{j=1}^{n-1} \left[\sum_{k_j=0}^{\infty} (-1)^{k_j} q^{k_j(k_j-1)/2 - \beta_j k_j} \begin{bmatrix} \beta_j \\ k_j \end{bmatrix}_q \right] \\ &\qquad \times \Phi_D\Big(a; b, q^{k_i - \beta_i}; q^{\alpha - m + 1}; q; x, y_i\Big) \end{aligned} \tag{5.25}$$

with α and β_i, $i = 1, \ldots, n-1$, positive integers.

Finally, let us stress that the relations we have derived are just a few examples of the many identities that can be obtained from the quantum algebra interpretation of the q-Appell and q-Lauricella functions that we have described. Our aim was to show the usefulness and simplicity of this approach, without claiming to be exhaustive.

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